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Minimization of AND-EXOR Expressions for Symmetric Functions

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SUMMARY This paper deals with minimization of ESOPs (exclusive-or sum-of-products) which represent symmetric functions. We propose an efficient simplification algorithm for symmetric functions, which guarantees the minimality for some subclass of symmetric functions, and present the minimum ESOPs for all 6-variable symmetric functions.

key words: *logic synthesis, AND-EXOR expression, symmetric function, logic minimization algorithm*

1. Introduction

AND-EXOR expressions usually require fewer products than AND-OR ones[8]. An AND-EXOR expression such that arbitrary product terms are combined with EXOR operators is called an exclusive-or sum-of-products expression (ESOP). Some minimization and simplification algorithms of ESOPs have been proposed [6], [7], [9], [10]. However, there are no efficient algorithms to minimize ESOPs for functions which have six or more variables. The simplification algorithms are more practical than minimization ones, but their results are not always minimum.

Since symmetric functions are basic arithmetic functions, their minimization and simplification have been also studied [1], [4], [6], [8]. Koda and Sasao gave minimum ESOPs for some symmetric functions [4], but for some 6-variable symmetric functions, their minimum ESOPs remain unknown.

In this paper, we propose an efficient algorithm to simplify ESOPs for symmetric functions, which guarantees the minimality for some subclass of symmetric functions, and we present the minimum ESOPs of all 6-variable symmetric functions, which are obtained by this algorithm.

2. Definitions and Basic Properties

The definitions and properties used throughout this paper are given in this section.

Definition 1: Product terms combined with Exclusive-OR operators form an *Exclusive-or Sum-Of-Products expression* (*ESOP*). The number of product terms in an ESOP F is denoted by $\tau(F)$. Among all ESOPs that represent a function f, those with a minimum number of product terms are called *minimum ESOPs* of f. The number of product terms in a minimum ESOP of f is denoted by $\tau(f)$.

Definition 2: Let f be an n-variable function with the input variables $X = (x_1, x_2, \cdots, x_n)$. For a permutation π on X, $\pi(f)$ denotes the function given by $f(\pi(x_1), \pi(x_2), \cdots, \pi(x_n))$. If $\pi(f) = f$ for any permutation π on X, f is said to be *symmetric.* An n-variable function g is *P-equivalent* to f if there exists a permutation π such that $\pi(f) = g$. The class of all functions P-equivalent to f is called *P-equivalence class* of f.

From the definitions, the following properties hold.

Property 1: Let f and g be functions with the input variables X, and π a permutation on X. Then $\tau(\pi(f)) = \tau(f)$ and $\pi(f \oplus g) = \pi(f) \oplus \pi(g)$ hold.

It is obvious that the above equations hold since π permutes only the variables of the function.

Property 2: Let f be a symmetric function and g be a function. Then, for any P-equivalent function $\pi(g)$ of g, $\tau(f \oplus g) = \tau(f \oplus \pi(g))$.

Proof: From Property 1 and the definition of symmetric functions, we have $\tau(f \oplus g) = \tau(\pi(f \oplus g)) = \tau(\pi(f) \oplus \pi(g)) = \tau(f \oplus \pi(g))$. □

Definition 3: For a function f and a variable x, the subfunctions of f with $x = 0$ and $x = 1$ are denoted by $f_{x:\{0\}}$ and $f_{x:\{1\}}$, respectively. $f_{x:\{0,1\}}$, which is also called the subfunction of f, is defined to be $f_{x:\{0\}} \oplus f_{x:\{1\}}$ and $f_{x:\{\}}$ denotes the logical zero function 0. If the both subfunctions $f_{x:\{0\}}$ and $f_{x:\{1\}}$ are symmetric, f is said to be x*-symmetric.*

An arbitrary symmetric function f is x-symmetric for any $x \in X$ since its subfunctions $f_{x:\{0\}}$ and $f_{x:\{1\}}$ are also symmetric, but x-symmetric functions are not always symmetric. Note that if f is x-symmetric, the subfunction $f_{x:\{0,1\}}$ is symmetric as well as $f_{x:\{0\}}$ and $f_{x:\{1\}}$.

3. Minimization Theorem of x-Symmetric Functions

Nishitani and Shimizu [6] gave the minimization theorem for arbitrary functions, which characterizes the number of product terms of minimum ESOPs. This theorem allows us to construct a minimization algorithm

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for arbitrary functions straightforwardly. However, in order to minimize the ESOP for an n-variable function, the algorithm must compute the minimum ESOPs of all $(n-1)$-variable functions, so it works only for functions with at most 5 variables, practically.

In this section, we give a similar theorem limited to x-symmetric functions. From it, we can also obtain a minimization algorithm for x-symmetric functions. Since it does not need to compute minimum ESOPs for all $(n-1)$-variable functions, it is a faster algorithm. The following is the minimization theorem given in [6].

Theorem 1 (Minimization Theorem): Let $\mathcal{F}^{n-1}$ be the class of all $(n-1)$-variable functions. For an n-variable function f and a variable x, the following equation holds.

$$\tau(f) = \min_{g \in \mathcal{F}^{n-1}} \{\tau(f_{x:\{0\}} \oplus g) + \tau(f_{x:\{1\}} \oplus g) + \tau(g)\}$$

Note that $f_{x:\{0\}} \oplus g$, $f_{x:\{1\}} \oplus g$, and g are $(n-1)$-variable functions. Since a minimum ESOP of a 1-variable function is easily obtained, it is possible to compute a minimum ESOP of an arbitrary n-variable function by applying the equation in Theorem 1 recursively.

However, as described above, this algorithm must minimize the functions, $f_{x:\{0\}} \oplus g$, $f_{x:\{1\}} \oplus g$, and g, for all $(n-1)$-variable functions $g \in \mathcal{F}^{n-1}$. The cardinality of $\mathcal{F}^{n-1}$ is $2^{2^{n-1}}$. This makes it practically impossible to minimize functions with more than 5 variables. Then we try to reduce the class by limiting to x-symmetric functions and obtain the following theorem.

Theorem 2 (Minimization Theorem for x-Symmetric Functions): Let $\mathcal{P}^{n-1}$ be the class of the representative functions of all P-equivalence classes of $(n-1)$-variable functions. For an n-variable x-symmetric function f, the following equation holds.

$$\tau(f) = \min_{g \in \mathcal{P}^{n-1}} \{\tau(f_{x:\{0\}} \oplus g) + \tau(f_{x:\{1\}} \oplus g) + \tau(g)\}$$

Proof: From Theorem 1, there exists an $(n-1)$-variable function g' such that $\tau(f) = \tau(f_{x:\{0\}} \oplus g') + \tau(f_{x:\{1\}} \oplus g') + \tau(g')$. Let g be the representative function of the P-equivalence class including g' and let π be a permutation such that $\pi(g) = g'$. Then, since $f_{x:\{0\}}$ and $f_{x:\{1\}}$ are symmetric, the following equations are obtained from Property 2: $\tau(f_{x:\{0\}} \oplus g) = \tau(f_{x:\{0\}} \oplus \pi(g)) = \tau(f_{x:\{0\}} \oplus g')$, $\tau(f_{x:\{1\}} \oplus g) = \tau(f_{x:\{1\}} \oplus \pi(g)) = \tau(f_{x:\{1\}} \oplus g')$, and $\tau(g) = \tau(\pi(g)) = \tau(g')$. Hence, we have $\tau(f) = \tau(f_{x:\{0\}} \oplus g') + \tau(f_{x:\{1\}} \oplus g') + \tau(g') = \tau(f_{x:\{0\}} \oplus g) + \tau(f_{x:\{1\}} \oplus g) + \tau(g)$. □

Since any symmetric function is x-symmetric, this theorem allows us to construct a minimization algorithm for symmetric functions, that is, a minimum ESOP for a symmetric function can be obtained by computing minimum ESOPs of $f_{x:\{0\}} \oplus g$, $f_{x:\{1\}} \oplus g$, and g only for every $g \in \mathcal{P}^{n-1}$. Note that minimum ESOPs for $f_{x:\{0\}} \oplus g$, $f_{x:\{1\}} \oplus g$, and g are obtained from the algorithm based on Theorem 1 since they are not symmetric functions in general. The algorithm based on Theorem 2 is faster than that based on Theorem 1 because $|\mathcal{P}^{n-1}| \leqq |\mathcal{F}^{n-1}|$. However, the cardinality of $\mathcal{P}^{n-1}$ is not enough small to minimize symmetric functions with 6 variables.

To derive a more efficient algorithm for x-symmetric functions, we generalize the equation in Theorem 2. Let f be an n-variable function. We have the following four expansions for an arbitrary $(n-1)$-variable function g.

$$\begin{aligned} f &= \bar{x}(f_{x:\{0\}} \oplus g) \oplus x(f_{x:\{1\}} \oplus g) \oplus (f_{x:\{\}} \oplus g) \\ &= \bar{x}(f_{x:\{\}} \oplus g) \oplus x(f_{x:\{0,1\}} \oplus g) \oplus (f_{x:\{0\}} \oplus g) \\ &= \bar{x}(f_{x:\{0,1\}} \oplus g) \oplus x(f_{x:\{\}} \oplus g) \oplus (f_{x:\{1\}} \oplus g) \\ &= \bar{x}(f_{x:\{1\}} \oplus g) \oplus x(f_{x:\{0\}} \oplus g) \oplus (f_{x:\{0,1\}} \oplus g) \end{aligned}$$

For each expansion above, we have the inequality $\tau(f) \leqq \sum_{R \subseteqq B,\ R \neq S} \tau(f_{x:R} \oplus g)$, where $S \subseteqq B = \{0,1\}$. Hence, $\tau(f) \leqq \min_{S \subseteqq B} \{\sum_{R \subseteqq B,\ R \neq S} \tau(f_{x:R} \oplus g)\}$. Then, we have the following corollary.

Corollary 1 Let $B = \{0,1\}$. For an n-variable x-symmetric function f, the following equation holds.

$$\tau(f) = \min_{g \in \mathcal{P}^{n-1}} \{\min_{S \subseteqq B} \{\sum_{R \subseteqq B,\ R \neq S} \tau(f_{x:R} \oplus g)\}\} \quad (1)$$

The minimization algorithm based on the above equation is less efficient than that based on Theorem 2 because it minimizes four $(n-1)$-variable functions for each g. However, in the next section we derive an efficient algorithm from this equation.

4. Simplification of x-Symmetric Functions

The computing time of the algorithm based on Theorem 2 depends mainly on the number of functions in $\mathcal{P}^{n-1}$. Although $|\mathcal{P}^{n-1}| \leqq |\mathcal{F}^{n-1}|$, with increasing n, it requires very large computing time practically, and it is difficult for this algorithm to minimize a symmetric function with 6 or more variables. Therefore, we attempt to reduce the class $\mathcal{P}^{n-1}$ to some subclass.

In [2], an approach to reduce the class $\mathcal{F}^{n-1}$ to the subclass $\mathcal{T}_k^{n-1}$ was presented. $\mathcal{T}_k^{n-1}$ denotes a class of all $(n-1)$-variable functions f with $\tau(f) \leqq k$, where k is a nonnegative integer. The algorithm with $\mathcal{T}_k^{n-1}$ is denoted by $A[\mathcal{T}_k]$. In general, it is a simplification algorithm, i.e., it does not always guarantee the minimality because $\mathcal{T}_k^{n-1} \subseteqq \mathcal{F}^{n-1}$. However, the algorithm $A[\mathcal{T}_k]$ has the following properties, where $\sigma_k(f)$ denotes the number of products of an ESOP for f obtained by $A[\mathcal{T}_k]$.

1. $\sigma_k(f) = \tau(f)$ if $\tau(f) < 3(k+1)$.
2. $\sigma_k(f) = \tau(f)$ if $\sigma_k(f) \leqq 3(k+1)$.

The above approach can be applied to $\mathcal{P}^{n-1}$. Let $\mathcal{P}_k^n$ be a class of all n-variable functions f in $\mathcal{P}^n$ such that $\tau(f) \leqq k$, where k is a nonnegative integer. By substituting $\mathcal{P}_k^{n-1}$ and $\sigma_k(f_{x:R}\oplus g)$ for $\mathcal{P}^{n-1}$ and $\tau(f_{x:R}\oplus g)$ in the Eq. (1) of Corollary 1 respectively, we define ρ_k as follows.

$$\rho_k(f) = \min_{g\in\mathcal{P}_k^{n-1}}\{\min_{S\subseteq B}\{\sum_{R\subseteq B,\ R\neq S}\sigma_k(f_{x:R}\oplus g)\}\}$$

From the equation ρ_k we can compute ESOPs for x-symmetric functions if all functions in $\mathcal{P}_k^{n-1}$ are generated efficiently or pre-computed. The simplification algorithm based on ρ_k is denoted by $A[\mathcal{P}_k]$. Note that $A[\mathcal{P}_k]$ calls $A[\mathcal{T}_k]$ to obtain ESOPs for $(n-1)$-variable functions $f_{x:R}\oplus g$ because they are not symmetric functions in general. The simplification algorithm $A[\mathcal{P}_k]$ is faster than $A[\mathcal{T}_k]$ because $|\mathcal{P}_k^{n-1}| \leqq |\mathcal{T}_k^{n-1}|$. As well as $A[\mathcal{T}_k]$, $A[\mathcal{P}_k]$ guarantees the minimality for some subclass of x-symmetric functions.

Theorem 3 Let k be a nonnegative integer and f be an x-symmetric function. $\rho_k(f)$ has the following properties.

1. $\rho_k(f) = \tau(f)$ if $\tau(f) < 3(k+1)$
2. $\rho_k(f) = \tau(f)$ if $\rho_k(f) \leqq 3(k+1)$

Theorem 3 can be proved in a similar way to the theorem of $\sigma_k(f)$ proved in [2]. The value $\rho_k(f)$ is the number of products of ESOPs obtained by the simplification algorithm $A[\mathcal{P}_k]$. Therefore, the property 1 of Theorem 3 shows that $A[\mathcal{P}_k]$ computes a minimum ESOP for any x-symmetric function f such that $\tau(f) < 3(k+1)$, and the property 2 guarantees that if the number of product terms of the ESOP obtained by $A[\mathcal{P}_k]$ is at most $3(k+1)$, it is minimum.

5. Minimum ESOPs for 6-Variable Symmetric Functions

The algorithm $A[\mathcal{P}_k]$ was implemented in C language and it was applied to all 6-variable symmetric functions on Sparc Station 10 (CPU: SuperSparc 50 MHz). The program obtained their minimum ESOPs. In this implementation, minimum ESOPs for 5-variable functions, which may not be symmetric, are computed by the simplification algorithm $A[\mathcal{T}_k]$. The classes $\mathcal{P}_k^5$ were pre-computed.

Table 1 shows the computing time of $A[\mathcal{P}_k]$ $(k \leqq 4)$ for a 6-variable symmetric function in the worst case. For 7-variable symmetric functions, $A[\mathcal{P}_k]$ with $k \leqq 1$ is available. Table 2 shows minimum ESOPs for 6-variable symmetric functions. In Table 2, the representative functions of symmetric-L-equivalence classes [4], [5] are presented and the other functions are omitted. The functions are written in value vectors. For an n-variable symmetric function f, let $v_i(f)$ be the output value of f on the input such that the number of variables with value 1 is equal to i. The value vector of f is the vector of $v_i(f)$'s: $[v_0(f), v_1(f), \cdots, v_n(f)]$. Nishitani and Shimizu [6] gave the table of $\tau(f)$ of all 5-variable symmetric functions and Koda and Sasao [4], [5] presented the results for 6-variable symmetric functions obtained by the simplification algorithm EXMIN2, which does not guarantee the minimality. The minimality of the ESOPs in Table 2 is guaranteed.

Our algorithm $A[\mathcal{P}_k]$ with $k = 4$ computed the minimum ESOPs for all 6-variable symmetric functions except three functions [0,1,1,0,1,1,0], [1,1,0,1,1,0,1], and [1,0,1,1,0,1,1]. Minimum ESOPs for the three functions were obtained by $A[\mathcal{P}_5]$ though not all functions in $\mathcal{P}_5^5$ were checked. Since ESOPs for these functions f obtained by $A[\mathcal{P}_4]$ have 16 products, we can conclude that $\tau(f) \geqq 15$ from the property 1 of Theorem 3. On the other hand, since $A[\mathcal{P}_5]$ obtains their ESOPs with 15 products when checking functions in $\mathcal{P}_5^5$, we have $\tau(f) \leqq 15$. Therefore these three ESOPs with 15 products are minimum.

Table 1 Computing time for a 6-variable symmetric function in the worst case (second).

$A[\mathcal{P}_0]$	$A[\mathcal{P}_1]$	$A[\mathcal{P}_2]$	$A[\mathcal{P}_3]$	$A[\mathcal{P}_4]$
0.039	0.21	4.4	115.2	2591.3

6. Conclusion

In this paper, we characterized minimum ESOPs for x-symmetric functions in terms of their subfunctions, and presented the algorithm $A[\mathcal{P}_k]$ with parameter k to minimize or simplify ESOPs of x-symmetric functions. This algorithm guarantees that it computes a minimum ESOP for an arbitrary symmetric functions f with $\tau(f) < 3(k+1)$ and that a simplified ESOP is minimum when the number of product terms in it is at most $3(k+1)$. Minimum ESOPs for all 6-variable symmetric functions were obtained by this algorithm.

References

[1] S. Even, I. Kohavi, and A. Paz, "On minimal modulo-2 sums of products for switching functions," IEEE Trans. Electron. Comput., vol.EC-16, pp.671–674, Oct. 1967.
[2] T. Hirayama and Y. Nishitani, "A simplification algorithm of AND-EXOR expressions guaranteeing minimality for some class of logic functions," IEICE Trans., vol.J78-D-I, no.4, pp.409–415, April 1995.
[3] N. Koda and T. Sasao, "A minimization method for AND-EXOR expressions using lower bound theorem," IEICE Trans., vol.J76-D-I, no.1, pp.1–10, Jan. 1993.
[4] N. Koda and T. Sasao, "A simplification method of ESOPs for symmetric functions," IPSJ SIG Notes, DA-74-5, pp.33–40, March 1995.
[5] N. Koda and T. Sasao, "An upper bound on the number of products in minimum ESOPs," IFIP WG10.5 Workshop on Application of Reed-Muller Expansion in Circuit Design, pp.94–101, Aug. 1995.
[6] Y. Nishitani and K. Shimizu, "Lower bounds on size of periodic functions in exclusive-OR sum-of-products expressions," IEICE Trans. Fundamentals, vol.E77-A, no.3,

Table 2 Minimum ESOPs for 6-variable symmetric functions.

$\tau(f)$	function	minimum ESOP
0	[0,0,0,0,0,0,0]	$= 0$
1	[1,0,0,0,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1$
2	[1,1,1,1,1,1,0]	$= x_6x_5x_4x_3x_2x_1 \oplus 1$
3	[0,1,1,1,1,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6x_5x_4x_3x_2x_1 \oplus 1$
6	[0,1,0,0,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_2 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1$
	[1,1,0,0,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_2x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1$
7	[1,0,1,0,1,0,0]	$= x_6x_5x_4x_3x_2x_1 \oplus x_6 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus \bar{x}_1$
	[1,0,0,0,0,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5x_4x_3x_2x_1 \oplus x_6\bar{x}_5x_4x_3x_2x_1 \oplus x_6x_5\bar{x}_4x_2x_1 \oplus x_6x_5x_4x_3x_2 \oplus x_6x_5x_4x_3x_1 \oplus x_6x_5\bar{x}_3x_2x_1$
8	[0,0,1,0,1,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus \bar{x}_1$
	[1,0,1,1,1,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_2x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6x_5x_4x_3x_2x_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus 1$
11	[0,0,1,0,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4x_3\bar{x}_2x_1 \oplus \bar{x}_6x_5x_4\bar{x}_3x_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_4\bar{x}_2 \oplus \bar{x}_6\bar{x}_3\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_3\bar{x}_2 \oplus x_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5\bar{x}_4x_3x_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3\bar{x}_2x_1$
	[1,0,1,0,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5x_4x_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4x_3\bar{x}_2 \oplus \bar{x}_6x_5x_4\bar{x}_3\bar{x}_1 \oplus \bar{x}_6\bar{x}_4\bar{x}_2x_1 \oplus \bar{x}_6\bar{x}_3x_2\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_3\bar{x}_2 \oplus x_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5\bar{x}_4x_3x_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3\bar{x}_2x_1$
12	[0,0,0,1,0,0,0]	$= \bar{x}_6x_5x_4x_1 \oplus \bar{x}_6x_5x_3 \oplus \bar{x}_6x_4x_2 \oplus \bar{x}_6x_3x_2x_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_2 \oplus x_6\bar{x}_4\bar{x}_3 \oplus x_6\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3x_2\bar{x}_1 \oplus \bar{x}_5x_4x_3\bar{x}_2x_1 \oplus x_5\bar{x}_4\bar{x}_3x_2x_1 \oplus x_5\bar{x}_4x_3\bar{x}_2\bar{x}_1$
	[0,1,0,1,0,0,0]	$= x_6x_5\bar{x}_4x_2x_1 \oplus x_6x_5x_4x_3x_2 \oplus x_6x_5x_4x_3x_1 \oplus x_6x_5\bar{x}_3x_2x_1 \oplus x_6x_4x_3x_2x_1 \oplus x_6 \oplus x_5x_4x_3x_2x_1 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus x_1$
	[0,0,1,1,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_2 \oplus \bar{x}_6\bar{x}_4\bar{x}_3 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6x_5x_2 \oplus x_6x_4x_3 \oplus x_6x_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_5x_4x_1 \oplus x_5x_3 \oplus x_4x_2 \oplus \bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_3x_2x_1$
	[1,1,0,0,0,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_2x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5x_4x_3x_2x_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6\bar{x}_5x_4x_3x_2x_1 \oplus x_6x_5\bar{x}_4x_2x_1 \oplus x_6x_5x_4x_3x_2 \oplus x_6x_5x_4x_3x_1 \oplus x_6x_5\bar{x}_3x_2x_1$
	[0,1,1,0,0,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_2 \oplus \bar{x}_6\bar{x}_4\bar{x}_3 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6x_5x_2 \oplus x_6x_4x_3 \oplus x_6\bar{x}_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_5x_4x_1 \oplus x_5\bar{x}_3 \oplus \bar{x}_4\bar{x}_2 \oplus \bar{x}_3x_2\bar{x}_1 \oplus x_3\bar{x}_2x_1$
	[1,0,1,0,1,1,0]	$= x_6x_5\bar{x}_4x_2x_1 \oplus x_6x_5x_4x_3\bar{x}_2\bar{x}_1 \oplus x_6x_5x_4x_3 \oplus x_6x_5\bar{x}_3x_2x_1 \oplus x_6x_4x_3x_2x_1 \oplus x_6 \oplus x_5x_4x_3x_2x_1 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus \bar{x}_1$
	[1,0,1,0,0,0,1]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5x_4x_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4x_3\bar{x}_2 \oplus \bar{x}_6x_5x_4\bar{x}_3\bar{x}_1 \oplus \bar{x}_6\bar{x}_4\bar{x}_2x_1 \oplus \bar{x}_6\bar{x}_3x_2\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_3\bar{x}_2 \oplus x_6x_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6x_5x_4x_3x_2x_1 \oplus \bar{x}_5\bar{x}_4x_3x_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3\bar{x}_2x_1$
	[1,1,1,0,0,0,1]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5x_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_2 \oplus \bar{x}_6x_5\bar{x}_3x_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_4x_3\bar{x}_2x_1 \oplus \bar{x}_6x_4\bar{x}_3\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_3\bar{x}_2 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5\bar{x}_4x_3x_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3\bar{x}_2x_1$
13	[1,0,0,1,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5x_4x_1 \oplus \bar{x}_6x_5x_3 \oplus \bar{x}_6x_4x_2 \oplus \bar{x}_6x_3x_2x_1 \oplus x_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_5\bar{x}_2 \oplus x_6\bar{x}_4\bar{x}_3 \oplus x_6\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3x_2\bar{x}_1 \oplus \bar{x}_5x_4x_3\bar{x}_2x_1 \oplus x_5\bar{x}_4\bar{x}_3x_2x_1 \oplus x_5\bar{x}_4x_3\bar{x}_2\bar{x}_1$
	[1,1,0,1,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6x_5\bar{x}_4x_2x_1 \oplus x_6x_5x_4x_3x_2 \oplus x_6x_5x_4x_3x_1 \oplus x_6x_5\bar{x}_3x_2x_1 \oplus x_6x_4x_3x_2x_1 \oplus x_6 \oplus x_5x_4x_3x_2x_1 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus x_1$
	[1,0,1,1,0,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_2 \oplus \bar{x}_6\bar{x}_4\bar{x}_3 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6x_5x_2 \oplus x_6x_4x_3 \oplus x_6x_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_5x_4x_1 \oplus x_5x_3 \oplus x_4x_2 \oplus \bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_3x_2x_1$
	[0,1,1,0,1,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_2x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6 \oplus x_6x_5x_4x_3x_2x_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus \bar{x}_1$
	[1,1,1,0,1,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_2 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6 \oplus x_6x_5x_4x_3x_2x_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_5 \oplus x_4 \oplus x_3 \oplus x_2 \oplus \bar{x}_1$
	[1,1,1,0,0,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_2 \oplus \bar{x}_6\bar{x}_4\bar{x}_3 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6x_5x_2 \oplus x_6x_4x_3 \oplus x_6\bar{x}_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_1 \oplus x_5x_4x_1 \oplus x_5\bar{x}_3 \oplus \bar{x}_4\bar{x}_2 \oplus \bar{x}_3x_2\bar{x}_1 \oplus x_3\bar{x}_2x_1$
14	[0,1,0,0,1,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_3x_1 \oplus \bar{x}_6\bar{x}_5x_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_6x_5x_4x_3x_2 \oplus \bar{x}_6x_5x_2x_1 \oplus \bar{x}_6x_4x_3\bar{x}_2x_1 \oplus x_6\bar{x}_5x_2 \oplus x_6x_5x_4\bar{x}_2x_1 \oplus x_6x_5x_3\bar{x}_2 \oplus x_6\bar{x}_4\bar{x}_1 \oplus x_6\bar{x}_3x_2\bar{x}_1 \oplus \bar{x}_5\bar{x}_4x_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5x_4x_3x_1 \oplus x_5\bar{x}_4\bar{x}_3\bar{x}_1 \oplus \bar{x}_4\bar{x}_3x_2$
	[1,1,0,0,1,0,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_1 \oplus \bar{x}_6\bar{x}_5x_4x_3x_2x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2 \oplus x_6x_5\bar{x}_4\bar{x}_3 \oplus x_6x_5\bar{x}_4\bar{x}_2x_1 \oplus x_6x_5x_4x_3\bar{x}_2\bar{x}_1 \oplus x_6x_5\bar{x}_3x_2\bar{x}_1 \oplus x_6x_4x_1 \oplus x_6x_3x_2 \oplus x_5x_4x_1 \oplus x_5x_3x_2 \oplus \bar{x}_4\bar{x}_3\bar{x}_1 \oplus \bar{x}_4x_2\bar{x}_1 \oplus x_4\bar{x}_3\bar{x}_2x_1$
	[1,1,1,0,1,1,0]	$= \bar{x}_6\bar{x}_5x_4x_3\bar{x}_2 \oplus \bar{x}_6\bar{x}_5\bar{x}_3x_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_3x_2 \oplus \bar{x}_6\bar{x}_4\bar{x}_2x_1 \oplus x_6\bar{x}_5\bar{x}_4x_3x_2 \oplus x_6x_5x_4\bar{x}_3\bar{x}_2 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6x_5x_3\bar{x}_1 \oplus x_6x_4x_2\bar{x}_1 \oplus \bar{x}_5\bar{x}_4x_1 \oplus x_5x_2\bar{x}_1 \oplus x_4x_3\bar{x}_1 \oplus \bar{x}_3\bar{x}_2x_1 \oplus 1$
	[1,1,0,0,1,0,1]	$= \bar{x}_6\bar{x}_5\bar{x}_4x_3x_2x_1 \oplus \bar{x}_6\bar{x}_5\bar{x}_4 \oplus \bar{x}_6\bar{x}_5\bar{x}_3\bar{x}_2 \oplus x_6x_5\bar{x}_4\bar{x}_3\bar{x}_2 \oplus x_6x_5x_4x_3\bar{x}_1 \oplus x_6x_5x_4x_2\bar{x}_1 \oplus x_6x_3x_2\bar{x}_1 \oplus x_6x_1 \oplus \bar{x}_5x_1 \oplus x_5x_3x_2\bar{x}_1 \oplus \bar{x}_4\bar{x}_3\bar{x}_1 \oplus \bar{x}_4x_2 \oplus x_4x_3\bar{x}_2x_1 \oplus \bar{x}_3x_2x_1$
15	[0,1,1,0,1,1,0]	$= \bar{x}_6\bar{x}_5\bar{x}_4 \oplus \bar{x}_6x_5x_4x_3x_2x_1 \oplus \bar{x}_6x_5\bar{x}_3x_2\bar{x}_1 \oplus \bar{x}_6x_4\bar{x}_3\bar{x}_2x_1 \oplus \bar{x}_6x_3\bar{x}_2\bar{x}_1 \oplus x_6\bar{x}_5x_3x_1 \oplus x_6\bar{x}_5x_2 \oplus x_6x_5x_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus x_6\bar{x}_4x_3x_2 \oplus x_6\bar{x}_4x_1 \oplus \bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_5x_4\bar{x}_3\bar{x}_1 \oplus x_5\bar{x}_4\bar{x}_3\bar{x}_2 \oplus x_5x_4 \oplus x_3x_2x_1$
	[1,1,0,1,1,0,1]	$= \bar{x}_6\bar{x}_5\bar{x}_4\bar{x}_2\bar{x}_1 \oplus \bar{x}_6\bar{x}_5x_4\bar{x}_3\bar{x}_1 \oplus \bar{x}_6x_5\bar{x}_4\bar{x}_3\bar{x}_2 \oplus \bar{x}_6x_5x_4 \oplus \bar{x}_6x_3x_2x_1 \oplus x_6\bar{x}_5\bar{x}_4 \oplus x_6x_5x_4x_3x_2x_1 \oplus x_6x_5\bar{x}_3x_2\bar{x}_1 \oplus x_6x_4\bar{x}_3\bar{x}_2x_1 \oplus x_6x_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_5x_3x_1 \oplus \bar{x}_5x_2 \oplus x_5x_4\bar{x}_3\bar{x}_2\bar{x}_1 \oplus \bar{x}_4x_3x_2 \oplus \bar{x}_4x_1$

pp.475–482, March 1994.

[7] M. Perkowski and M. Chrzanowska-Jeske, "An exact algorithm to minimize mixed-radix exclusive sums of products for incompletely specified Boolean functions," Proc. Int. Symposium on Circuits and Systems '90, pp.1652–1655, 1990.

[8] T. Sasao and P. Besslich, "On the complexity of mod-2 sum PLA's," IEEE Trans. Comput., vol.C-39, no.2, pp.262–266, Feb. 1990.

[9] T. Sasao, "EXMIN2: A simplification algorithm for exclusive-or sum of products expressions for multiple-valued-input two-valued-output functions," IEEE Trans. Comput.-Aided Des. Integrated Circuits & Syst., vol.12, no.5, pp.621–632, May 1993.

[10] T. Sasao, "An exact minimization of AND-EXOR expressions using BDDs," IFIP WG10.5 Workshop on Application of Reed-Muller Expansion in Circuit Design, Sept. 1993.